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# ON FUNCTIONAL EQUATIONS OF PREHOMOGENEOUS ZETA DISTRIBUTIONS OVER A LOCAL FIELD OF CHARACTERISTIC $p$

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ABSTRACT. For a local field of characteristic 0, the functional equations of zeta distributions of prehomogeneous vector spaces are obtained by M.Sato, T.Shintani, J.Igusa and F.Sato ( See [17], [9], [13], [15] ). In this paper, we shall consider the case of local fields of characteristic  $p > 0$ .

## §1. $K$ -regular P.V.'s

We fix a local field  $K$  of characteristic  $p > 0$ . Let  $G$  be a connected linear algebraic group,  $\rho$  its rational representation of  $G$  on a finite-dimensional vector space  $V$ , all defined over an algebraic closure  $\bar{K}$  of  $K$ . We call a triplet  $(G, \rho, V)$  a prehomogeneous vector space (abbrev. P.V.) if  $V$  has a Zariski-dense  $G$ -orbit  $Y$ .

Any point of  $Y$  is called a generic point and the isotropy subgroup

$G_y = \{g \in G; \rho(g)y = y\}$  of a generic point  $y$  is called a generic isotropy subgroup. Note that we have  $\dim G_y = \dim G - \dim V$  if and only if  $y \in Y$ . A non-zero rational function  $f(x)$  on  $V$  is called a relative invariant of  $(G, \rho, V)$  if  $f(\rho(g)x) = \chi(g)f(x)$  holds for any  $g \in G$  and  $x \in Y$  where  $\chi : G \rightarrow GL_1$  is a rational character of  $G$ .

The complement  $S$  of  $Y$  is a Zariski-closed set which is called the singular set of the P.V.  $(G, \rho, V)$ . Now we assume that  $(G, \rho, V)$  is defined over  $K$ , i.e.,  $G, \rho, V$  are all defined over  $K$ . Let  $S_i = \{x \in V; f_i(x) = 0\}$  ( $i = 1, \dots, l$ ) be the  $K$ -irreducible component of the  $K$ -rational points  $S_K$  of  $S$  of codimension one defined by a  $K$ -irreducible (not necessarily absolutely irreducible) polynomial  $f_i(x)$  ( $i = 1, \dots, l$ ). Then  $f_1(x), \dots, f_l(x)$  are algebraically independent relative invariants and any relative invariant  $f(x)$  in  $K(V)$  is of the form  $f(x) = c \cdot f_1(x)^{m_1} \cdots f_l(x)^{m_l}$  ( $c \in K^\times, (m_1, \dots, m_l) \in \mathbb{Z}^l$ ). We call  $f_1(x), \dots, f_l(x)$  the basic  $K$ -relative invariants of  $(G, \rho, V)$ . Let  $\chi_i$  be the rational character of  $G$  corresponding to  $f_i$  ( $i = 1, \dots, l$ ). Let  $X(G)_K$  be the group of  $K$ -rational characters of  $G$ ,  $X_1(G)_K$  its subgroup corresponding to  $K$ -relative invariants. Then  $X_1(G)_K$  is a free abelian group of rank  $l$  generated by  $\chi_1, \dots, \chi_l$ .

Let  $G_1$  be a subgroup of  $G$  generated by the commutator subgroup  $[G, G]$  and a generic isotropy subgroup. This does not depend on a choice of a generic point. For  $\chi \in X(G)_K$ , it is in  $X_1(G)_K$  if and only if  $\chi|_{G_1} = 1$ . For a relative invariant  $f(x)$  of  $(G, \rho, V)$ , we can define a rational map  $\varphi_f : Y \rightarrow V^*$  by

$$\varphi_f(x) = {}^t \left( \frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_1}(x), \dots, \frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_n}(x) \right)$$

where  $V^*$  is the dual vector space of  $V$ . We sometimes denote  $\varphi_f(x)$  by  $\text{grad log } f(x)$ . By a direct calculation, we have

(1)  $\varphi_f(\rho(g)x) = \rho^*(g)\varphi_f(x)$  for  $g \in G$  and  $x \in Y$  where  $\rho^*$  denotes the contra-gradient representation of  $\rho$ ,

and

(2)  $\langle d\rho(A)x, \varphi_f(x) \rangle = \delta\chi(A)$  for  $x \in Y$  and  $A \in \text{Lie}(G)$  where  $d\rho$  ( resp.  $\delta\chi$  ) is the infinitesimal representation of  $\rho$  ( resp. the infinitesimal character of  $\chi$  ) of the Lie algebra  $\text{Lie}(G)$  of  $G$ .

A relative invariant  $f(x)$  is called non-degenerate if  $\varphi_f : Y \rightarrow V^*$  is dominant and the Hessian  $H_f(x) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$  is not identically zero. In this case, a rational function  $F(x) = \frac{f(x)^n}{H_f(x)}$  ( $n = \dim V$ ) is a relative invariant corresponding to the character  $\chi_0(g) = \det \rho(g)^2$ .

If there exists a non-degenerate relative invariant  $f(x)$  in  $K(V)$ , we say that  $(G, \rho, V)$  is a  $K$ -regular P.V. Then we have  $\det \rho(g)^2 \in X_1(G)_K$ . In general, we denote by  $Y_K, S_K$ , etc.  $K$ -rational points of  $Y, S$ , etc. We write  $X_1^*(G)_K$  ( resp.  $X^*(G)_K, Y^*, S^*$ , etc.) for  $(G, \rho^*, V^*)$  which corresponds to  $X_1(G)_K$  ( resp.  $X(G)_K, Y, S$ , etc. ) for  $(G, \rho, V)$ .

### Proposition 1.1

Assume that  $(G, \rho, V)$  and  $(G, \rho^*, V^*)$  are  $K$ -regular P.V.'s. Then we have the following assertion.

(1)  $X_1(G)_K = X_1^*(G)_K$ .

(2) For a non-degenerate  $K$ -relative invariant  $f$ , the map  $\varphi = \text{grad log } f : Y \rightarrow Y^*$  is bijective.

[Proof]

Since  $\varphi(Y)$  is a Zariski-dense  $G$ -orbit in  $V^*$ , we have  $\varphi(Y) = Y^*$ , i.e.,  $\varphi =$  surjective. Since  $\rho^*(g)\varphi(x) = \varphi(\rho(g)x)$ , we have  $G_x \subset G_{\varphi(x)}$  for  $x \in Y$ . Now let  $f^*$  be a non-degenerate relative invariant in  $K(V^*)$ , and put  $\varphi^* = \text{grad log } f^* : Y^* \rightarrow Y$ . Similarly we have  $G_y \subset G_{x'}$  for  $y = \varphi(x)$  and  $x' = \varphi^*(y)$ , and hence  $G_x \subset G_y \subset G_{x'}$ . Since  $x' = \rho(g_0)x$  for some  $g_0 \in G$ , we have  $G_{x'} = g_0 G_x g_0^{-1} \supset G_x$ . Since  $\dim G_{x'} = \dim G_x$ , the algebraic group  $G_{x'}$  and  $G_x$  have the same connected component  $H$  of the identity. Since  $G_{x'}$  is isomorphic to  $G_x$ , the numbers of their connected components coincide, i.e.,  $[G_{x'} : H] = [G_x : H]$  with  $G_{x'} \supset G_x$ . This implies  $G_{x'} = G_x$ , and hence  $G_x = G_y$  with  $y = \varphi(x)$ .

Thus we have  $G_1 = G_1^*$  and hence  $X_1(G)_K = X_1^*(G)_K$ . Note that  $X_1(G)_K = \{\chi \in X(G)_K; \chi|_{G_1} = 1\}$ . Now assume that  $\varphi(x_1) = \varphi(x_2)$  with  $x_2 = \rho(g)x_1$  for some  $g \in G$ . Then we have  $\varphi(x_1) = \varphi(x_2) = \varphi(\rho(g)x_1) = \rho^*(g)\varphi(x_1)$  and hence  $g \in G_{\varphi(x_1)} = G_{x_1}$ , i.e.,  $x_2 = \rho(g)x_1 = x_1$ . Thus  $\varphi$  is injective.  $\square$

Now assume that  $(G, \rho, V)$  is a  $K$ -regular P.V. Then, as we have seen above, the dual triplet  $(G, \rho^*, V^*)$  is a P.V. For a generic point  $y \in Y^*$ , a dominant morphism  $\psi : G \rightarrow V^*$  defined by  $\psi(g) = \rho^*(g)y$  is called an open orbit morphism.

### Proposition 1.2

Assume that  $(G, \rho, V)$  is a  $K$ -regular P.V. and an open orbit morphism  $\psi : G \rightarrow V^*$  is a separable morphism. Then there exists a  $K$ -relative invariant  $f^*$  such that  $\text{grad log } f^* : Y^* \rightarrow V$  is dominant.

[Proof]

Let  $f$  be a non-degenerate relative invariant in  $K(V)$  and put  $\varphi = \text{grad log } f : Y \rightarrow Y^*$ . First we show that  $\varphi$  is injective. Assume that  $\varphi(x) = \varphi(x')$ . Since  $\delta\chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$ , we have  $\langle x - x', d\rho^*(A)\varphi(x) \rangle = 0$  for all  $A \in \text{Lie}(G)$ . Since  $\psi : G \rightarrow V^*$  with  $\psi(g) = \rho^*(g)\varphi(x)$  is separable, we have  $\{d\rho^*(A)\varphi(x); A \in \text{Lie}(G)\} = V^*$ , and hence  $x - x' = 0$ , i.e.,  $x = x'$ . For any  $g \in G_{\varphi(x)} (\supset G_x)$ , we have  $\varphi(\rho(g)x) = \rho^*(g)\varphi(x) = \varphi(x)$ . As  $\varphi$  is injective, we have  $\rho(g)x = x$ , i.e.,  $g \in G_x$ . This implies that  $G_x = G_{\varphi(x)}$  and hence  $X_1(G)_K = X_1^*(G)_K$ . A rational character  $\chi$  corresponding to  $f$  is in  $X_1(G)_K$  and hence  $\chi^{-1} \in X_1^*(G)_K$ . This implies that there exists a relative invariant  $f^*$  in  $K(V^*)$  satisfying  $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$  for  $g \in G$  and  $y \in Y^*$ .

Put  $\varphi^* = \text{grad log } f^*$ . Then we have  $\langle \varphi^*(y), d\rho^*(A)y \rangle = -\delta\chi(A)$ . Since  $\delta\chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$ , we have

$$\langle x - \varphi^*(y), d\rho^*(A)y \rangle = 0 \text{ for } y = \varphi(x) \text{ and all } A \in \text{Lie}(G).$$

Since the open orbit morphism  $\psi$  is separable, we have

$$\{d\rho^*(A)y; A \in \text{Lie}(G)\} = V^*,$$

and hence  $\varphi^*(y) = x \in Y$ , i.e.,  $\varphi^*(Y^*) = Y$ .  $\square$

Note that in the case of  $\text{ch}(K) = 0$ , the proof of Proposition 1.2 gives the equivalence between  $K$ -regularity of  $(G, \rho, V)$  and that of  $(G, \rho^*, V^*)$ .

### Proposition 1.3

Assume that  $(G, \rho, V)$  and  $(G, \rho^*, V^*)$  are  $K$ -regular P.V.'s. Then we have  $\# \rho(G)_K \backslash Y_K = \# \rho^*(G)_K \backslash Y_K^*$ .

[Proof]

Let  $f$  be a non-degenerate relative invariant in  $K(V)$  and put  $\varphi = \text{grad log } f$ . Then for any  $x \in Y_K$ , we have

$\varphi(\rho(G)_K \cdot x) = \rho^*(G)_K \cdot \varphi(x) \subset Y_K^*$ , i.e.,  $\varphi$  maps an orbit in  $Y_K$  to an orbit in  $Y_K^*$ .

By Proposition 1.1, this map  $\varphi$  is injective, and hence  $\# \rho(G)_K \backslash Y_K \leq \# \rho^*(G)_K \backslash Y_K^*$ . Similarly we have  $\# \rho^*(G)_K \backslash Y_K^* \leq \# \rho(G)_K \backslash Y_K$ .  $\square$

Now we shall consider a sufficient condition that  $\# \rho(G)_K \backslash Y_K$  is finite.

Professor J.P. Serre kindly let us know the following theorem with the proof which was explained by Tits to him.

#### Theorem 1.4

Let  $K$  be a local field of characteristic  $p > 0$  ( or more generally let  $K$  be a field complete with respect to a discrete valuation, and with the residue field  $k$  of type (F) in the sense of Serre [18]. Let  $G$  be a connected smooth reductive group over  $K$ . Then  $H^1(K, G)$  is finite.

[Proof] ( after Serre's letter on 9th. September 1992. )

Let  $K'$  be the maximal unramified extension of  $K$ . The field  $K'$  is known to be of  $\dim. \leq 1$  ( in the sense of CG, II, §3 ). By a theorem of Steinberg ( for  $K'$  perfect ) and of Borel-Springer ( for  $K'$  imperfect - see Borel Col. Papers II, p.761 ) we have  $H^1(K', G) = 0$ . Hence the Galois cohomology of  $G$  over  $K$  is killed by  $K'$ , i.e., it is equal to  $H^1(K'/K, G)$ . We may now apply a theorem of Bruhat-Tits ( J.Fac.Sci.Tokyo, 34 (1987), p.693, th.3.12 ); this says that  $H^1(K'/K, G)$  is contained in a finite union of cohomology sets  $H^1(k, G_i)$ , where the  $G_i$ 's are algebraic linear groups ( non necessarily connected ) over  $k$ . Since  $k$  is type (F), each  $H^1(k, G_i)$  is finite ( see e.g. Borel, Col.Papers II, p.404, th.6.2, or Coh. Gal. III-30, th.4 ). Hence  $H^1(K, G)$  is finite.

$\square$

#### Proposition 1.5

Let  $(G, \rho, V)$  be a P.V. defined over  $K$  with a reductive generic isotropy subgroup. Then  $\# \rho(G)_K \backslash Y_K$  is finite.

[Proof]

Let  $H$  be a generic isotropy subgroup of a point in  $Y_K$ . Then there exists a bijection between  $\rho(G)_K \backslash Y_K$  and  $\text{Ker}(H^1(K, H) \rightarrow H^1(K, G))$  ( see Serre [18] ). By Theorem 1.4,  $H^1(K, H)$  is finite, and hence  $\rho(G)_K \backslash Y_K$  is a finite set.

$\square$

#### Example 1.6

Let  $G$  be the subgroup of  $GL_n$  consisting of all lower triangular matrices. Let  $V$  be the totality of symmetric  $n \times n$  matrices and define  $\rho$  by

$$\rho(g)x = gx^tg \text{ for all } g \in G$$

and  $x \in V$ . Since  $\dim G = \dim V$ , a generic isotropy subgroup is a finite subgroup and hence we have  $\# \rho(G)_K \backslash Y_K = \nu < +\infty$  by Proposition 1.5.

Moreover  $\det x$  is a non-degenerate  $K$ -relative invariant. By  $\text{tr}(xy)$  ( $x, y \in V$ ), we identify  $V$  with its dual  $V^*$ .

Then  $(G, \rho, V)$  and  $(G, \rho^*, V^*)$  are  $K$ -regular P.V.'s. Hence, by Proposition 1.3, we have  $\# \rho^*(G)_K \backslash Y_K^* = \nu < +\infty$ .

### Proposition 1.7

Let  $(G, \rho, V)$  be an irreducible regular P.V. defined over  $K$ . Then we have  $\# \rho(G)_K \backslash Y_K < +\infty$ .

[Proof]

By a classification of irreducible P.V.'s ( see Z.Chen [4] ), we know that a generic isotropy subgroup is reductive.

□

## §2. Zeta distributions

Let  $K$  be a local field of characteristic  $p > 0$ . Assume that  $(G, \rho, V)$  and its dual  $(G, \rho^*, V^*)$  are  $K$ -regular P.V.'s. Moreover we shall assume that  $Y_K = Y_1 \cup \dots \cup Y_\nu$  decomposes into a finite union of  $\rho(G)_K$ -orbits  $Y_i$  ( $1 \leq i \leq \nu$ ), i.e.,  $\# \rho(G)_K \backslash Y_K = \nu < +\infty$ . Then by Proposition 1.3, we have  $Y_K^* = Y_1^* \cup \dots \cup Y_\nu^*$ .

Let  $f_1(x), \dots, f_l(x)$  ( resp.  $f_1^*(y), \dots, f_l^*(y)$  ) be basic  $K$ -relative invariants of  $(G, \rho, V)$  ( resp.  $(G, \rho^*, V^*)$  ). Let  $\chi_i$  ( resp.  $\chi_i^*$  ) be the corresponding character of  $f_i$  ( resp.  $f_i^*$  ). Then we have

$$X_1(G)_K = \langle \chi_1, \dots, \chi_l \rangle \text{ and } X_1^*(G)_K = \langle \chi_1^*, \dots, \chi_l^* \rangle.$$

By Proposition 1.1, we have  $X_1(G)_K = X_1^*(G)_K$  so that there exists uniquely a matrix

$$U = (u_{ij}) \in GL_l(\mathbf{Z})$$

satisfying  $\chi_i = \prod_{j=1}^l \chi_j^{*u_{ij}}$ . Since  $\det \rho(g)^2 \in X_1(G)_K$ , we have  $\det \rho(g)^2 = \chi_1^{2\lambda_1} \dots \chi_l^{2\lambda_l}$  for some  $\lambda = (\lambda_1, \dots, \lambda_l) \in (\frac{1}{2} \mathbf{Z})^l$  and  $\det \rho^*(g)^2 = \chi_1^{*2\lambda_1^*} \dots \chi_l^{*2\lambda_l^*}$  for some  $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*) \in (\frac{1}{2} \mathbf{Z})^l$ . Since  $\det \rho^*(g) = \det \rho(g)^{-1}$ , we have  $\lambda^* = -\lambda U$ .

### Example 2.1

For simplicity, we deal with the case  $n = 2$  in Example 1.6. Then we have

$$G = \{g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}; ab \neq 0\}$$

and

$$V = \{X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}\}.$$

The basic  $K$ -relative invariants of  $(G, \rho, V)$  ( resp.  $(G, \rho^*, V^*)$  ) are  $f_1(X) = x$  and  $f_2(X) = \det X$  ( resp.  $f_1^*(X) = z$ ,  $f_2^*(X) = \det X$  ) corresponding to  $\chi_1(g) = a^2$ ,  $\chi_2(g) = a^2 b^2$  ( resp.  $\chi_1^*(g) = b^{-2}$ ,  $\chi_2^*(g) = a^{-2} b^{-2}$  ) for

$$g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$$

in  $G$ .

Hence  $\chi_1 = \chi_1^* \chi_2^{*-1}$  and  $\chi_2 = \chi_2^{*-1}$  so that we have

$$U = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Since

$$\det \rho \left( \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \right) = a^3 b^3,$$

we have  $\lambda = \lambda^* = (0, \frac{3}{2})$ .

Let  $\{\varepsilon_1, \dots, \varepsilon_\nu\}$  be the complete representatives of  $K^\times / K^{\times 2}$  in  $K^\times$ . Then we have  $Y_K = Y_1 \cup \dots \cup Y_\nu$  with

$$Y_i = \{y \in Y_K; f_2(y) \equiv \varepsilon_i \text{ mod } K^{\times 2}\} \quad (i = 1, \dots, \nu).$$

Let  $\omega^{(i)} : K^\times \rightarrow \mathbb{C}^\times$  ( $i = 1, \dots, l$ ) be a quasicharacter, i.e., a continuous homomorphism.

For  $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$  and the basic  $K$ -relative invariants  $f(x) = (f_1(x), \dots, f_l(x))$ , we write  $\omega(f(x))$  instead of  $\prod_{i=1}^l \omega^{(i)}(f_i(x))$  for simplicity of notations.

Let  $|\cdot|$  be the absolute value of  $K$  normalized by  $|\pi| = q^{-1}$  for a prime element  $\pi$  where  $q$  is the module of  $K$ . For  $s = (s_1, \dots, s_l)$ , we write  $\omega_s = (|\cdot|^{s_1}, \dots, |\cdot|^{s_l})$  so that  $\omega_s(f(x)) = \prod_{i=1}^l |f_i(x)|^{s_i}$ .

Let  $dx$  be the Haar measure on  $V_K = K^n$  normalized by  $\int_{R^n} dx = 1$  where  $R$  is the maximal compact subring of  $K$ . Since  $d(\rho(g)x) = |\det \rho(g)| dx$  and  $\omega_\lambda(f(\rho(g)x)) = |\det \rho(g)| \omega_\lambda(f(x))$ , the measure  $d_Y(x) = \frac{dx}{\omega_\lambda(f(x))}$  is a  $G$ -invariant measure on  $Y$ .

For  $\Phi \in \mathfrak{S}(V_K)$  where  $\mathfrak{S}(V_K)$  denotes the Schwartz-Bruhat space of  $V_K$ , we define an integral

$$Z_i(\omega, \Phi) = \int_{Y_i} \omega(f(x)) \Phi(x) d_Y(x) \quad (i = 1, \dots, \nu).$$

Now any quasi-character  $\omega^{(i)} : K^\times \rightarrow \mathbb{C}^\times = \{z \in \mathbb{C}; z \neq 0\}$  can be written uniquely as  $\omega^{(i)} = |\cdot|^{s_i} \cdot \phi_i$  for some  $s_i \in \mathbb{C}$  and  $\phi_i : R^\times \rightarrow \mathbb{C}_1^\times = \{z \in \mathbb{C}; |z| = 1\}$  where  $R^\times$  is the units of  $R$ . Put  $\text{Re } \omega^{(i)} = \text{Re } s_i$  ( $i = 1, \dots, l$ ). The following lemma is easy to prove and we omit the proof ( cf. F.Sato [15] ).

## Lemma 2.2

If  $\operatorname{Re} \omega^{(i)} > \lambda_i$  ( $i = 1, \dots, l$ ), the integral  $Z_i(\omega, \Phi)$  is absolutely convergent and holomorphic with respect to  $s = (s_1, \dots, s_l) \in (\mathbf{C}/(\frac{2\pi i}{\log q} \mathbf{Z}))^l \cong \mathbf{C}^{\times l}$  for  $\omega = (|s_1 \cdot \phi_1, \dots, |s_l \cdot \phi_l)$ .

Let  $\mathfrak{S}'(V_K) = \{z : \mathfrak{S}(V_K) \rightarrow \mathbf{C}, \text{ } \mathbf{C} - \text{linear mapping}\}$  be the space of distributions on  $V_K$ . By Lemma 2.2, the mapping  $\Phi \mapsto Z_i(\omega, \Phi)$  defines a distribution on  $V_K$  when  $\operatorname{Re} \omega^{(i)} > \lambda_i$  ( $i = 1, \dots, l$ ).

For  $(G, \rho^*, V^*)$ , we can define similar distribution  $Z_j^*(\omega)$  ( $j = 1, \dots, \nu$ ) given by

$$Z_j^*(\omega, \Phi^*) = \int_{Y_j^*} \omega(f^*(y)) \Phi^*(y) d_{Y^*}(y).$$

Now we fix a non-trivial additive character  $\psi : K \rightarrow \mathbf{C}_1^\times$  and define the Fourier transformation  $\mathfrak{S}(V_K^*) \ni \Phi^* \mapsto \hat{\Phi}^* \in \mathfrak{S}(V_K)$  by

$$\hat{\Phi}^*(x) = \int_{V_K^*} \Phi^*(y) \psi(\langle x, y \rangle) dy$$

where  $dy$  is a Haar measure on  $V_K^*$  dual to a fixed Haar measure on  $V_K$ .

For  $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$ , put  $\omega^* = \omega^U = (\prod_{i=1}^l \omega^{(i)u_{i1}}, \dots, \prod_{i=1}^l \omega^{(i)u_{il}})$ .

Our purpose is to show that  $Z_i(\omega)$  and  $Z_j^*(\omega)$  are continued analytically to all  $\omega$  and satisfy the functional equation:

$$(2.1) \quad \hat{Z}_i(\omega) = \sum_{j=1}^{\nu} \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}) \quad (i = 1, \dots, \nu)$$

under some additional conditions where

$\hat{Z}_i(\omega)(\Phi^*) = Z_i(\omega, \hat{\Phi}^*)$ . Recall that  $\omega_{\lambda^*} = (|_{\lambda_1^*}, \dots, |_{\lambda_l^*})$  with  $\det \rho^*(g)^2 = \chi_1^{2\lambda_1^*} \cdots \chi_l^{2\lambda_l^*}$ .

Actually when  $K$  is a local field of  $ch(K) = 0$ , then (2.1) is obtained under some conditions and it is called "the fundamental theorem of P.V. over  $K$ ".

### §3. Rationality for almost all $p$

For a rational prime  $p$ , let  $K_p$  denotes the local field with the constant field  $\mathbf{F}_p$ . For  $f \in \mathbf{Z}[x_1, \dots, x_n]$ , we denote  $f \bmod p \in \mathbf{F}_p[x_1, \dots, x_n]$  by  $f_p$ . Then we have the following theorem which is suggested by Professor M.Kashiwara.



**Theorem 3.1**

For almost all  $p$ , the integral

$$Z_p(s, \Phi_p) = \int_{K_p^n} |f_p(x)|_{K_p}^s \Phi_p(x) d_p x$$

is a rational function of  $t = p^{-s}$  where  $\Phi_p \in \mathfrak{S}(K_p^n)$  and  $d_p x$  is a Haar measure on  $K_p^n$ .

[Proof]

Let  $K = \mathbb{Q}((t))$  be a field of formal power series over  $\mathbb{Q}$ ,  $X = \Omega^n$  the affine space and  $X_K = K^n$ . Let  $f$  denote the morphism  $X \rightarrow \Omega$  defined by  $f(x)$ ; then there exists a nonsingular algebraic variety  $Y$  and a projective morphism  $h : Y \rightarrow X$  both defined over  $K$  with the following property : let  $b$  denote an arbitrary point of  $Y_K$ ,  $\mathfrak{O}_K$  the local ring of  $Y$  at  $b$  relative to  $K$  (consisting of "functions" defined over  $K$ ), and  $\mathfrak{M}_K$  the ideal of non-units of  $\mathfrak{O}_K$ ; then there exists an ideal basis  $(y_1, \dots, y_n)$  of  $\mathfrak{M}_K$ , elements  $u, v$  of  $\mathfrak{O}_K - \mathfrak{M}_K$ , and integers  $N_i \geq 0, \nu_i \geq 1$  for  $1 \leq i \leq n$  such that

$$f \circ h = u \cdot \prod_{i=1}^n y_i^{N_i}, \quad h^*(dx) = v \cdot \prod_{i=1}^n y_i^{\nu_i-1} dy.$$

The existence of such a pair  $(Y, h)$  is guaranteed by Hironaka's theorem [5] p.109 -p.326]. Then for almost all  $p$ , the reduction modulo  $p$  is well-defined and we have similar results for  $K_p, f_p, \dots$  etc. Then by just similar argument as in Appendix of Igusa [11], we obtain our result.  $\square$

**Remark 3.2**

Let  $K$  be a number field. For  $f \in \mathfrak{O}_K[x_1, \dots, x_n]$ , we have a similar result as Theorem 3.1 for almost all prime ideals  $\mathfrak{P}$  of  $\mathfrak{O}_K$ .

**§4. Functional equations****Lemma 4.1**

Let  $G$  denote a locally compact totally disconnected group,  $H$  a closed subgroup of  $G$ ,  $X = H \backslash G$ , and  $\omega : G \rightarrow \mathbb{C}^\times$  a quasicharacter. Put

$$\xi_X(\omega) = \{T \in \mathfrak{S}(X)'; gT = \omega(g)^{-1}T \text{ for all } g \in G\}.$$

Then we have  $\dim_{\mathbb{C}} \xi_X(\omega) \leq 1$ . Moreover  $\dim_{\mathbb{C}} \xi_X(\omega) = 1$  if and only if  $\Delta_G \cdot \omega|_H = \Delta_H$  where  $\Delta_G, \Delta_H$  denotes the module of  $G, H$  respectively.

[Proof]

See Igusa [9] p.1015.  $\square$

Let  $(G, \rho, V)$  and its dual  $(G, \rho^*, V^*)$  be  $K$ -regular P.V.'s with

$$\# \rho(G)_K \backslash Y_K = \nu < +\infty$$

where  $K$  is a local field of characteristic  $p$ . Then, by Proposition 1.3, we have

$$Y_K = Y_1 \cup \dots \cup Y_\nu \text{ and } Y_K^* = Y_1^* \cup \dots \cup Y_\nu^* \text{ i.e., } \# \rho^*(G)_K \backslash Y_K^* = \nu.$$

As in §2, we can define the zeta distribution  $Z_i(\omega, \Phi)$  ( resp.  $Z_i^*(\omega, \Phi^*)$  ) which is convergent when  $\operatorname{Re} \omega^{(j)} > \lambda_j$  ( resp.  $\operatorname{Re} \omega^{(j)} > \lambda_j^*$  ) (  $1 \leq i \leq \nu, 1 \leq j \leq l$  ).

We denote by  $Z_i(\omega)$  the distribution defined by  $\Phi \mapsto Z_i(\omega, \Phi)$  etc.

### Proposition 4.2

We have

$$(1) Z_j^*(\omega^* \omega_{\lambda^*}) \in \xi_{Y_j^*}(\omega^* \omega_{\lambda^*})$$

and

$$(2) \hat{Z}_i(\omega) \in \xi_{Y_j^*}(\omega^* \omega_{\lambda^*}).$$

$$(i, j = 1, \dots, \nu)$$

[Proof]

By a direct calculation, we obtain our results.  $\square$

### Proposition 4.3

Let  $K$  be a local field of characteristic  $p > 0$  with the module  $q$ . For  $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$  with  $\omega^{(i)} = \omega_{s_i} \cdot \phi_i$  (  $\phi_i(\pi) = 1$  for a prime element  $\pi$  ), assume that  $Z_i(\omega, \Phi)$  and  $Z_j^*(\omega, \Phi^*)$  are rational functions of  $q^{-s_1}, \dots, q^{-s_l}$ . Then for all  $\Phi^* \in \mathfrak{S}(Y_K^*)$ , we have

$$Z_i(\omega, \hat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)$$

for  $i, j = 1, \dots, \nu$ .

[Proof]

Since  $Z_i(\omega, \Phi)$  and  $Z_j(\omega, \Phi^*)$  are rational functions, it is defined for all  $\omega$  except poles and hence by Lemma 4.1 and Proposition 4.2, we have our result.  $\square$

### Theorem 4.4

Let  $(G, \rho, V)$  be a  $K$ -regular P.V. satisfying the following conditions:

(C1) its dual  $(G, \rho^*, V^*)$  is a  $K$ -regular P.V. such that

$$\# \rho^*(G)_K \backslash V_K^* < +\infty,$$

(C2) for  $x \in S_K^*$ , there exists  $\chi \in X_1(G)_K$  satisfying  $\chi(G_{x,K}) \not\subseteq R^\times$  where  $R^\times$  is the units of the maximal compact subring  $R$  of  $K$

and

(C3)  $Z_j(\omega, \Phi)$  is a rational function of  $q^{-s_1}, \dots, q^{-s_l}$  where

$$\omega = (\omega^{(1)}, \dots, \omega^{(l)}) \text{ with } \omega^{(i)} = \omega_{s_i} \text{ ( } 1 \leq i \leq l \text{ )}.$$

Then we have the functional equation

$$Z_i(\omega, \hat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)$$

for all  $\Phi^* \in \mathfrak{S}(V_K^*)$  for  $i, j = 1, \dots, \nu$  where  $\nu = \# \rho^*(G)_K \backslash Y_K^*$ .

[Proof]

The condition (C2) corresponds to Lemma 2.2 in F.Sato [15] p474 for the case of  $ch(K) = 0$ . Then the proof is just similar as the case of  $ch(K) = 0$  ( using Proposition 4.3 ) ( See Igusa [9] and F.Sato [15] p.477 ).  $\square$

Now let  $(G, \rho, V)$  be a reductive  $\mathbf{Q}$ -regular P.V. Then for almost all  $p$ , we have a reduction modulo  $p$  and we obtain  $K_p$ -regular P.V.  $(G_p, \rho_p, V_p)$  where  $K_p$  is a local field with the constant field  $\mathbf{F}_p$ .

(Assumption A )

Assume that  $\# \rho_p(G)_{K_p} \backslash S_{K_p} < +\infty$  and for  $x \in S_{K_p}$ , there exists  $\chi \in X_1(G_p)_{K_p}$  satisfying  $\chi(G_{p,x,K_p}) \not\subseteq R_p^\times$  for almost all  $p$ .

Let  $(G, \rho, V)$  be a reductive  $\mathbf{Q}$ -regular P.V. with ( Assumption A ). Let  $f_1, \dots, f_l$  be basic  $\mathbf{Q}$ -relative invariants with  $\mathbf{Z}$ -coefficients. Denote  $|f_1 \bmod p|_{K_p}^{s_1^1} \dots |f_l \bmod p|_{K_p}^{s_l^1}$  by  $|f^{(p)}(x)|_{K_p}^s$  and

$$Z_i^p(s, \Phi_p) = \int_{(Y_{K_p})_i} |f^{(p)}(x)|_{K_p}^s \Phi_p(x) dY_p(x)$$

for  $\Phi_p \in \mathfrak{S}(V_{K_p})$ .

#### Theorem 4.5

Let  $(G, \rho, V)$  be a reductive  $\mathbf{Q}$ -regular P.V. with ( Assumption A ). Then for almost all rational prime  $p$ , the integral  $Z_i^p(s, \Phi_p)$  ( $i = 1, \dots, \nu_p$ ,  $Y_{K_p} = Y_1 \cup \dots \cup Y_{\nu_p}$ ) is a rational function and satisfies the functional equation:

.,

$$Z_i^p(s, \hat{\Phi}_p) = \sum_{j=1}^{\nu_p} \Gamma_{ij}(s) Z_j^p(s^*, \Phi_p)$$

(  $i = 1, \dots, \nu_p$  ).

When  $l = 1$ , we have  $s^* = \frac{n}{d} - s$  with  $n = \dim V$  and  $d = \deg f$ . In general, for  $\omega = \omega_s = \omega_{s_1} \dots \omega_{s_l}$ , we have  $\omega_{s^*} = \omega^* \omega_{\lambda^*}$ .

[Proof]

By Theorem 4.4 and using the results of §1 and §3, we obtain our result.  $\square$

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